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On Oscillation of Linear Differential Systems

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1

We are concerned with the system of differential equations

$$U''(t) + p(t)\,U(t) = 0 \qquad \text{on} \qquad [a, \infty), \tag{1.1}$$

where p is an $n \times n$ Hermitian matrix-valued function, the components of which are locally integrable. A *solution* U of (1.1) is an $n \times n$ matrix function, for which U' is absolutely continuous, and which satisfies (1.1) a.e. on $[a, \infty)$. For the discussion of the oscillatory behavior of solutions of (1.1) it is necessary to restrict our attention to the class of solutions, called *conjoined* solutions, which satisfy

$$U'^*(t)\,U(t) \equiv U^*(t)\,U'(t). \tag{1.2}$$

System (1.1) is said to be *oscillatory* on $[a, \infty)$ if there exists a conjoined solution $U(t)$ for which there is a sequence (t_n), $t_n \to \infty$, such that $\det U(t_n) = 0$, $n = 1, 2,\ldots$. It follows from the Sturm-type separation theorem of Morse [4] that if the system is oscillatory then every conjoined solution is singular on some sequence (t_n), $t_n \to \infty$. Thus the system is *nonoscillatory* if there exists a conjoined solution U for which $\det U(t) \neq 0$ for $t > \alpha \geqslant a$, for some α.

Wong [12], Willett [10], and others have made extensive studies of oscillation and nonoscillation of *scalar* differential equations of the form (1.1). In particular, Wong introduces a hierarchy of Riccati integral equations which he shows is useful in the study of (1.1), especially when the function p is allowed to oscillate. It is the purpose of this paper to show that, in part, the work of Willett and Wong

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can be extended to the matrix system (1.1). Of fundamental importance to the work of Wong is a theorem due to Wintner [11] later extended by Hartman [1]. In Section 3, we present an analog of that theorem, Theorem 3.1, and also develop some of the above-mentioned Riccati integral equation theory. In Section 4, we present a result, Theorem 4.3, for systems which extends a theorem due to Opial [5]. The proof of Theorem 4.3 is completely different from that of Opial's theorem, and in fact our theorem appears to extend Opial's theorem even in the scalar context.

2

The following results will be used to obtain the main theorems of this paper. Of fundamental importance to the study of oscillation theory is the basic Riccati inequality.

LEMMA 2.1. *System* (1.1) *is nonoscillatory if and only if there exists an* $n \times n$ *Hermitian matrix function* W *defined on* $[\alpha, \infty)$, *for some* $\alpha \geqslant a$, *satisfying*

$$W'(t) + W^2(t) + p(t) \leqslant 0 \qquad \text{a.e.} \quad on \quad [\alpha, \infty). \tag{2.1}$$

(Matrix inequalities are used in the sense of "nonnegative definite," "positive definite," etc.) This result goes back to Wintner [11] in the scalar case and appears in Sternberg [8] in the context of systems. Note that if (1.1) is non-oscillatory with a solution U satisfying (1.2) which is nonsingular on $[\alpha, \infty)$, then $W = U'U^{-1}$ is Hermitian and satisfies the Eq. (2.1).

LEMMA 2.2. *Let* W *be an* $n \times n$ *Hermitian matrix solution of the equality in* (2.1) *on* $[\alpha, \infty)$ *and suppose that*

$$\liminf_{t \to \infty} \pi^{i*} \left\{ (t - \alpha)^{-1} \int_\alpha^t \int_\alpha^s p(\sigma)\, d\sigma\, ds \right\} \pi^i > -\infty \tag{2.2}$$

is satisfied for some set of unit vectors $\pi^1, \pi^2, \pi^3, \ldots, \pi^n$ *spanning* C_n.

Then

$$\lim_{t \to \infty} (t - \alpha)^{-1} \int_\alpha^t \int_\alpha^s p(\sigma)\, d\sigma\, ds = C \tag{2.3}$$

and

$$\int_\alpha^\infty W^2(s)\, ds \tag{2.4}$$

exist as finite $n \times n$ *matrices and* W *satisfies*

$$W(t) = C - \int_\alpha^t p(s)\, ds + \int_t^\infty W^2(s)\, ds \qquad \text{on} \qquad [\alpha, \infty). \tag{2.5}$$

This result follows from the corresponding scalar theorem [1].[1] In particular, if $r = \pi^* W \pi$, $q = \pi^* p \pi$, then $r^2 \leqslant \pi^* W^2 \pi$ for $\| \pi \| = 1$ and hence $r' + r^2 + q \leqslant 0$ from which (2.3) follows.

LEMMA 2.3. (See, e.g., [2, 7].) *If* (1.1) *is nonoscillatory, there exists a conjoined solution U of* (1.1) *which is nonsingular on* $[\alpha, \infty)$, *for α sufficiently large, and which satisfies* $\int_\alpha^\infty U^{-1}(s)\, U^{*-1}(s)\, ds < +\infty$, *i.e., the integral converges finitely.*

3

Wintner [11], in the scalar case, shows that if (1.1) is nonoscillatory and if $\int^\infty p(s)\, ds$ exists and is finite, then

$$\int^\infty \exp \left\{ -2 \int^t \int_\tau^\infty p(s)\, ds\, d\tau \right\} dt = +\infty. \tag{3.1}$$

Hartman [1] shows that the "2" in (3.1) can be replaced by "4."

One of the main theorems of this paper provides an analog of the "improved" (3.1) for system (1.1). Under the assumption that (2.3) holds, i.e., the limit exists finitely and has the value C, put

$$P(t) = C - \int_a^t p(s)\, ds. \tag{3.2}$$

Note that if $\int_a^\infty p(s)\, ds$ exists, then $P(t) = \int_t^\infty p(s)\, ds$. Let $X(\cdot;\, T)$ be the solution of the first-order linear system

$$X'(t;\, T) = 2P(t)\, X(t;\, T), \qquad X(T;\, T) = E. \tag{3.3}$$

We now present the Wintner-type theorem.

THEOREM 3.1. *Let* (2.2) *hold and let* (1.1) *be nonoscillatory. Then*

$$\left| \int^\infty X^*(\cdot;\, \tau)\, X(\cdot;\, \tau)\, d\tau \right| = +\infty. \tag{3.4}$$

Proof. Let U be the solution provided in Lemma 2.3, and for t sufficiently large, $t \geqslant \alpha$, put $W(t) = U'(t)\, U^{-1}(t)$. Let $V(t) = \int_t^\infty W^2(s)\, ds$ then by (2.5) and (3.2), $W(t) = P(t) + V(t)$. Note that $0 \leqslant (V - P)^2 = V^2 - VP - PV + P^2$ implies $VP + PV \leqslant V^2 + P^2$ thus $V'(t) = -W^2 \leqslant -2(PV + VP)$. Multi-

[1] The authors wish to thank the referee for pointing this out and for making other suggestions regarding this paper.

plying appropriately by X and X^* yields $d\{X^*(t; s)\, V(t)\, X(t; s)\}/dt \leqslant 0$, and hence for $T < t$ we have $X^*(t; s)\, V(t)\, X(t; s) \leqslant X^*(T; s)\, V(T)\, X(T; s)$, which implies $V(t) \leqslant \gamma X^*(T; t)\, X(T; t)$, $t > T$, for γ some positive number. (Note that if $W(t) \equiv 0$ then $P(t) \equiv 0$ and the conclusion of the theorem holds trivially.)

If the conclusion of the theorem is false then $\int_T^\infty X^*(T; \tau)\, X(T; \tau)\, d\tau < \infty$ and and thus $\int_T^\infty V(\tau)\, d\tau < \infty$. It follows that $\int_T^\infty (s - T)\, W^2(s)\, ds < \infty$. Now $\int_T^\infty sW^2(s)\, ds < \infty$ implies $\int_T^\infty s[W^2(s)]_{ii}\, ds < \infty$, $i = 1, 2,..., n$, which yields

$$\int_T^\infty s \,\| W(s)\|^2\, ds < \infty, \tag{3.5}$$

since $W^2 \geqslant 0$ implies $\operatorname{tr} W^2(s) \geqslant \| W^2(s)\| = \| W(s)\|^2$.

Let $Z(\cdot; T)$ be the solution of the first-order linear system $Z' = W(t)\, Z$, $Z(T; T) = U(T)$. Then for each unit vector $\pi \in C_n$ we have

$$d\{\pi^* Z^*(t; T)\, Z(t; T)\, \pi\}/dt \leqslant 2 \,\| W(t)\|\, \pi^* Z^*(t; T)\, Z(t; T)\, \pi.$$

Integrating from T to t, $t > T$, and applying the Gronwall inequality we obtain

$$\pi^* Z^*(t; T)\, Z(t; T)\, \pi \leqslant \pi^* U^*(T)\, U(T)\, \pi \exp \left\{2 \int_T^t \| W(s)\|\, ds\right\},$$

and thus the corresponding matrix inequality holds.

Using the Schwarz inequality and (3.5) we have

$$\int_T^t \| W(s)\|\, ds \leqslant \left(\int_T^t ds/s\right)^{1/2} \left(\int_T^t s \,\| W(s)\|^2\, ds\right)^{1/2} \leqslant \{K \ln(t/T)\}^{1/2}, \qquad t > T,$$

where K is a positive number, and thus $Z^*(t; T)\, Z(t; T) \leqslant U^*(T)\, U(T) \times \exp\{2[K \ln(t/T)]^{1/2}\}$. Taking inverses, one has

$$Z^{-1}(T; t)\, Z^{*-1}(T; t) \geqslant U^{-1}(T)\, U^{*-1}(T) \exp\{-2[K \ln(t/T)]^{1/2}\}, \qquad t > T. \tag{3.6}$$

Note that $\int^\infty \exp\{-2[K \ln(t/T)]^{1/2}\}\, dt = \infty$ and thus the integral, with respect to t, of the left member of (3.6) is $+\infty$. But it is not difficult to show that $Z(t; T) \equiv U(t)$ and thus we have shown that $\int^\infty U^{-1}(t)\, U^{*-1}(t)\, dt = \infty$. This contradicts the first statement of the proof.

We remark that in the case of scalar equations (1.1) the previous proof reduces to that of Hartman [1]. Also, since for any $\pi \in C_n$, $\| \pi \| = 1$, (1.1) nonoscillatory implies $u'' + (\pi^* p(t)\, \pi)\, u = 0$ is nonoscillatory (see Remark following Lemma 2.2), it follows that a variant of Theorem 3.1 is obtainable directly from the corresponding scalar result, namely, by replacing (3.4) with

$$\int_\alpha^\infty \exp \left\{-4 \int_\alpha^\tau [\pi^* P(s)\, \pi]^+\, ds\right\} d\tau = \infty$$

(cf. 1973 version of [3]).

We now present a theorem on integral Riccati inequalities which relates in the scalar case to work of Wong [12] and Willett [9, 10]. Their work is based in part on a result of Wintner [11], later elaborated upon by Willett. As in Lemma 2.2, these authors place sufficient conditions on p so that if the corresponding scalar Riccati equation has a solution W then $\int^{\infty} W^2$ converges and Eq. (2.5) holds.

Here, we also assume (2.2) and, when (1.1) is nonoscillatory, that $P(t)$ is given by (3.2). Define, when the integral exists,

$$\bar{P}(t) = \int_t^{\infty} Y^*(s; t)\, P^2(s)\, Y(s; t)\, ds, \tag{3.7}$$

where $Y(\cdot; s)$ is the solution of the initial value problem $Y' = P(t)\, Y$; $Y(s) \doteq E$.

The following theorem essentially extends Willett's Theorem 5.4 [10] to systems (1.1). We remark that a stronger conclusion in Theorem 3.1 would yield the necessity of (3.9) for nonoscillation as well as its sufficiency.

THEOREM 3.2. *Let* (2.2) *hold.*

(i) *If* (1.1) *is nonoscillatory then* $\bar{P}(t)$ *exists finitely on* $[\alpha, \infty)$ *and there exists a Hermitian solution* V *of*

$$V(t) \geqslant \bar{P}(t) + \int_t^{\infty} Y^*(s; t)\, V^2(s)\, Y(s; t)\, ds \qquad \text{on} \qquad [\alpha, \infty). \tag{3.8}$$

(ii) *Let* $\bar{P}(t)$ *exist finitely and suppose the inequality*

$$V^2(t) \geqslant \left[\bar{P}(t) + \int_t^{\infty} Y^*(s; t)\, V^2(s)\, Y(s; t)\, ds\right]^2 \tag{3.9}$$

has a Hermitian solution V *on* $[\alpha, \infty)$; *then* (1.1) *is nonoscillatory.*

Proof. (i) Let $W(t)$ satisfy (2.5), and put $V(t) = \int_t^{\infty} W^2(s)\, ds$; then

$$d\{Y^*(t; \alpha)\, V(t)\, Y(t; \alpha)\}/dt = -Y^*(t; \alpha)\, \{P^2(t) + V^2(t)\}\, Y(t; \alpha).$$

Now integrating from t to τ, it follows that

$$V(t) = Y^*(\tau; t)\, V(\tau)\, Y(\tau; t) + \int_t^{\tau} Y^*(s; t)\, \{P^2(s) + V^2(s)\}\, Y(s; t)\, ds. \tag{3.10}$$

Since the left side of (3.10) is independent of τ, and $V(\tau) \geqslant 0$, it follows that $\bar{P}(t)$ and $\int_t^{\infty} Y^* V^2 Y\, ds$ exist finitely and that (3.8) is satisfied.

(ii) Let $V(t)$ satisfying (3.9) be given. Put

$$W(t) = P(t) + \bar{P}(t) + \int_t^{\infty} Y^*(s; t)\, V^2(s)\, Y(s; t)\, ds \equiv P(t) + H(t);$$

then $W'(t) = -p(t) - P(t)\, H(t) - H(t)\, P(t) - P^2(t) - V^2(t)$, and hence $W'(t) + W^2(t) + p(t) = H^2(t) - V^2(t) \leqslant 0$. The conclusion now follows from Lemma 2.1.

COROLLARY 3.3. *Let* (2.2) *hold. If there exists* $\pi \in C_n$, $\|\pi\| = 1$, *such that*

$$\pi^* \int_\alpha^\infty Y^*(s; \alpha)\, P^2(s)\, Y(s; \alpha)\, ds \pi = \infty$$

then (1.1) *is oscillatory.*

The corollary follows immediately from (i) above.

An additional corollary appears in Section 4.

4

Opial [5] shows, for scalar equations (1.1), that if $P(t) = \int_t^\infty p(s)\, ds$ exists finitely, then

$$\int_t^\infty P^2(s)\, ds \leqslant P(t)/4 \tag{4.1}$$

implies nonoscillation, and

$$\int_t^\infty P^2(s)\, ds \geqslant (1 + \epsilon)\, P(t)/4, \qquad \text{for some } \epsilon > 0, \tag{4.2}$$

implies oscillation. We develop some analogs of (4.1) in this section.

Reid [6] shows for systems (1.1) that $P^2(t) \leqslant p(t)/4$ implies nonoscillation. His method of proof easily extends to establish the following lemma.

LEMMA 4.1. *Let* q, $\hat{Q}$ *and* $\hat{P}$ *be* $n \times n$ *Hermitian matrix-valued functions satisfying* $\hat{P}'(t) = -p(t)$, $\hat{Q}'(t) = -q(t)$, *and* $(\hat{P}(t) + \hat{Q}(t))^2 \leqslant q(t)$ *on* $[\alpha, \infty)$. *Then* (1.1) *is nonoscillatory. In particular,* $4\hat{P}^2(t) \leqslant p(t)$ *implies nonoscillation.*

The main statement of the lemma follows on taking $W(t) = \hat{P}(t) + \hat{Q}(t)$ in (2.1), the secondary one on putting $q = p$ and $\hat{Q} = \hat{P}$.

The following lemma, for systems (1.1), is the same as the corresponding theorem for scalar equations, which is due to Hartman. (See, e.g., [3].) The proof is the same, using the fact that for $n \times n$ Hermitian matrices A and B,

$$AB + BA \leqslant A^2 + B^2. \tag{4.3}$$

LEMMA 4.2. *Let* $\hat{P}$ *be as in Lemma* 4.1. *If* $U''(t) + 4\hat{P}^2(t)\, U(t) = 0$ *is nonoscillatory then* (1.1) *is nonoscillatory.*

Opial's theorem relating to (4.1) may now be formulated.

THEOREM 4.3. *Let* $\hat{P}$ *be as in Lemma* 4.1 *and let* $\hat{\mathscr{P}}$ *be an* $n \times n$ *Hermitian matrix function satisfying* $\hat{\mathscr{P}}'(t) = -\hat{\mathscr{P}}^2(t)$. *Then*

$$\hat{\mathscr{P}}^2(t) \leqslant \hat{P}^2(t)/16 \qquad on \qquad [\alpha, \infty) \tag{4.4}$$

implies (1.1) *is nonoscillatory. In particular,* $(\int_t^\infty \hat{P}^2(s)\, ds)^2 \leqslant \hat{P}^2(t)/16$ *on* $[\alpha, \infty)$ *implies* (1.1) *is nonoscillatory.*

Proof. Let $p_1(t) = 4\hat{P}^2(t)$ then using (4.4) $\hat{P}_1(t) \equiv 4\hat{\mathscr{P}}(t)$ satisfies $4\hat{P}_1{}^2(t) \leqslant p_1(t)$ and hence, by Lemma 4.1, $U'' + p_1(t)\, U = 0$ is nonoscillatory, i.e., $U'' + 4\hat{P}^2(t)\, U = 0$ is nonoscillatory. The theorem now follows from Lemma 4.2.

A second Opial-type theorem is given by the following result.

THEOREM 4.4. *Let* (2.3) *hold. If* $\bar{P}(t)$ *in* (3.7) *exists finitely and*

$$\left[\int_t^\infty Y^*(s; t)\, \bar{P}^2(s)\, Y(s; t)\, ds\right]^2 \leqslant \bar{P}^2(t)/16 \qquad on \qquad [\alpha, \infty),$$

then (1.1) *is nonoscillatory.*

This theorem follows from part (ii) of Theorem 3.2. Choose $V(t) = 2\bar{P}(t)$ and apply the matrix inequality (4.3), to verify (3.9).

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